Schrodinger eqn. for Hydrogen atom ⇒

Schrodinger equation for an electron moving in three dimensions around a nucleus can be written as.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi + \frac{2m(E-V)\psi}{\hbar^2} = 0 \quad \text{--- (1)}$$

where ψ is the wavefunction of electron, m is mass, E is total energy, V is potential energy, $\hbar = \frac{h}{2\pi}$

The potential energy of an electron in a hydrogen atom is given by

$$V = \frac{-e^2}{4\pi\epsilon_0 r} \quad \text{--- (2)}$$

where $-e$ is the charge on electron and e is the charge on proton (hydrogen) and r is the distance between electron and proton or the radial distance.

$$r = \sqrt{x^2+y^2+z^2}$$

If we substitute the value of r by the above equation and the try to solve it the eqn. will become cumbersome so we will convert the cartesian co-ordinates x, y and z in terms of r in eqn. (1) and solve it

we will write the schrodinger equation in terms of spherical polar co-ordinates r, θ and ϕ.

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} + \frac{2m}{h^2}(E-V)\psi = 0$$

Multiplying both sides by $r^2\sin^2\theta$ we get

$$\sin^2\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial^2\psi}{\partial\phi^2} + \frac{2mr^2\sin^2\theta}{\hbar^2}\left(E + \frac{e^2}{4\pi\epsilon_0 r}\right)\psi = 0 \quad \text{--- (3)}$$

This eqn. will be solved by separation of variables

$$\psi(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$$

$$\frac{\partial \psi}{\partial r} = \Theta\Phi \frac{\partial R}{\partial r} = \Theta\Phi \frac{dR}{dr} \quad \text{——— (4)}$$

$$\frac{\partial \psi}{\partial \theta} = R\Phi \frac{\partial \Theta}{\partial \theta} = \Theta\Phi \frac{d\Theta}{d\theta} \quad \text{——— (5)}$$

$$\frac{\partial \psi}{\partial \phi} = R\Theta \frac{\partial \Phi}{\partial \phi} = R\Theta \frac{d\Phi}{d\phi} \quad \text{——— (6)}$$

$$\frac{\partial^2 \psi}{\partial \phi^2} = R\Theta \frac{d^2\Phi}{d\phi^2}$$

we will have to substitute 4, 5 and (6) in (3) and dividing throughout by $R\Theta\Phi$ we get

$$\frac{\Theta\Phi}{R\Theta\Phi}\sin^2\theta \frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{R\Phi}{R\Theta\Phi}\sin\theta \frac{\partial}{\partial \theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \frac{R\Theta}{R\Theta\Phi}\frac{\partial^2\psi}{\partial\phi^2} + \frac{2mr^2\sin^2\theta}{h^2}\left(E + \frac{e^2}{4\pi\epsilon_0 r}\right)\frac{R\Theta\Phi}{R\Theta\Phi} = 0$$

$$\frac{1}{R}\sin^2\theta \frac{d}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta}\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} + \frac{2mr^2\sin^2\theta}{h^2}\left(E + \frac{e^2}{4\pi\epsilon_0 r}\right) = 0$$

$$\frac{1}{R}\sin^2\theta \frac{d}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta}\sin\theta \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \frac{2mr^2\sin^2\theta}{h^2}\left(E + \frac{e^2}{4\pi\epsilon_0 r}\right) = -\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} \quad \text{——— (}$$

This equation is correct if both sides and equal the same constant and we presume that constant eq to m_l^2

$$-\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = m_l^2$$

multiplying both sides by Φ

$$\frac{d^2\Phi}{d\phi^2} + m_l^2\Phi = 0 \quad \text{——— (8)}$$